Analytical and Numerical Description of some Nonlinear Evolution Equations

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Abstract

In this paper, an analytical method with the aid of exp-function is used to obtain generalized travelling wave solutions of a Nonlinear Evolution Equation of variable coefficients. It is shown that the exp-function method, with the help of symbolic computation, provides a straightforward and powerful mathematical tool to solve such equations arises in mathematical physics.

Key Word: Exp-function, Travelling wave solution, Fisher Equation, Burger Equation

MSC-AMS: 35Q53, 35Q51, 37K10

1. INTRODUCTION

The investigation of exact solutions of Nonlinear Evolution Equation (NLEEs) plays an important role in the study of nonlinear physical or mathematical phenomena. The importance of obtaining the exact solutions of these nonlinear equations, if available, will facilitate the verification of numerical solvers and aids in the stability analysis of solutions. In the past several decades, many effective methods for obtaining exact solutions of NLEEs have been presented, such as the tanh-function method [1,2] extended tanh method [3,4], F-expansion method [5,6], sine-cosine method [7,8] Jacobian elliptic function method [9,10] homotopy perturbation method [11,12], variational iteration method [13,14] and Adomian method [15,16] and so on.

Recently, He and Wu [17] proposed exp-function method, to obtain generalized solitary solutions and periodic solutions. Applications of this method can be found in [18-20] for solving nonlinear evolution equations arising in physical sciences. The solution procedure of this method is very simple and can easily extend to other kinds of nonlinear evolution equations.

The present paper deals with the solution of the following Nonlinear Evolution Equation with variable coefficient with the help of exp-function method :

$$u_t - u_{xx} + \alpha(t) u u_x = \beta(t) u(1 - u), \tag{1}$$

where $\alpha(t)$, $\beta(t)$ are arbitrary functions of *t*. When $\alpha(t) = 0, \beta(t)$ is a arbitrary constant, equation (1) turns to Fisher equation

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$$u_t - u_{xx} = \beta u(1 - u), \tag{2}$$

Exact solution of equation (2) was found by Ablowitz and Zeppetella in [21] at $C_0 = \pm\frac{5}{\sqrt{6}}$.

When $\beta(t) = 0, \alpha(t)$ is a arbitrary constant then equation (1) turns to Burgers equation

$$u_t - u_{xx} + \alpha u u_x = 0 \tag{3}$$

which is used to describe the spread of sound wave in the medium with viscidity and heat exchange if we do not consider the medium's frequently dispersive character. The Burgers equations with variable coefficient can also be used to describe the cylindrical and spherical wave propagation in models such as over fall, traffic flow and so on.

2. ANALYTICAL SOLUTION

In order to obtain the solution of equation (1), we consider the transformation

$$u = u(\xi) \quad , \quad \xi = kx + \int \tau(t)\,dt \tag{4}$$

where k is a constant, $\tau(t)$ is an integrable function of **t** to be determined later, then equation (1) becomes an ordinary differential equation

$$\tau(t)u' + k\alpha(t)uu' - k^2u'' - \beta(t)u(1-u) = 0 \tag{5}$$

Where prime denotes the differential with respect to ξ

According to the Exp-function method, we assume that the solution of equation (5) can be expressed in the form

$$u(\xi) = \frac{\sum_{n=-c}^{d} a_n \exp(n\xi)}{\sum_{n=-p}^{q} b_m \exp(m\xi)} = \frac{a_{-c}e^{-c\xi} + \ldots + a_d e^{d\xi}}{b_{-p}e^{-p\xi} + \ldots + b_p e^{q\xi}} \tag{6}$$

Where c, d, p and q are positive integers which are unknown to be further determined, a_n and b_m are unknown constants.

In order to determine values of d and q, we balance the linear term of highest order in equation (5) with the highest order nonlinear term, and the linear term of lowest order in equation (5) with the lowest order nonlinear term, respectively. By simple calculation, we have

$$u''(\xi) = \frac{h_1 \exp[(d + 3q)\xi] + \ldots}{h_2 \exp[4q\xi] + \ldots} \tag{7}$$

and $$u(\xi)u'(\xi) = \frac{h_3 \exp[(2d + q)\xi] \;+\; \ldots}{h_4 \exp[3q\xi] \;+\; \ldots} = \frac{h_3 \exp[(2d + 2q)\xi] \;+\; \ldots}{h_4 \exp[4q\xi] \;+\; \ldots} \tag{8}$$

where h_i are to determined coefficient only for simplicity. Balancing highest order of Exp- function in equation (7) and (8) we have $d+3q=2d+2q$ so $d=q$ (9)

Similarly to determine values of c and p, we balance the linear term of lowest order in equation (5)

$$u''(\xi)=\frac{...+s_1\exp\left[-(c+3p)\xi\right]}{...+s_2\exp\left[-4p\xi\right]} \tag{10}$$

and $$u(\xi)u'(\xi)=\frac{s_3\exp\left[-(2c+p)\xi\right]\ +\ ...}{...+s_4\exp\left[-3q\xi\right]}=\frac{...+s_3\exp\left[-(2c+2p)\xi\right]}{...+s_4\exp\left[-4p\xi\right]} \tag{11}$$

Where s_i are determine coefficient only for simplicity. Balancing highest order of Exp- function in Eq. (10) and (11) we have $c+3p=2c+2p$; $c=p$ (12)

We can freely choose the values of c and d, but the final solution does not strongly depend upon the choice of values of c and d [19]. For simplicity, we set $b_1 = 1$, $p = c = 1$ and $d = q = 1$ equation (6) becomes

$$u(\xi)=\frac{a_1e^{\xi}+a_0+a_{-1}e^{-\xi}}{e^{\xi}+b_0+b_{-1}e^{-\xi}} \tag{13}$$

Substituting equation (13) into (5) we have

$$\frac{1}{A}\left[C_3e^{3\xi}+C_2e^{2\xi}+C_1e^{\xi}+C_0+C_{-1}e^{-\xi}+C_{-2}e^{-2\xi}+C_{-3}e^{-3\xi}\right]=0 \tag{14}$$

and

$$A = \left(exp\ (\xi)\ +\ b_0+b_{-1}exp(-\xi)\right)^3$$

$$C_3 = -a_1\beta(t)\ +\ a^2{}_1\beta(t)$$

$$C_2 = -2a_1b_0\beta(t)-ka_1a_0\alpha(t)+a^2{}_1b_0\beta(t)+a_1b_0\tau(t)+ka^2{}_1b_0\alpha(t)-a_0\tau(t)-a_0\beta(t)+k^2a_1b_0 -k^2a_0+2a_1a_0\beta(t)$$

$$C_1 = a^2{}_0\beta(t)-2ka_1a_{-1}\alpha(t)-a_0b_0\tau(t)+2a_0a_1b_0\beta(t)-k^2a_1b_0{}^2-ka_0^2\alpha(t)+2ka^2{}_1b_{-1}\alpha(t)- a_{-1}\beta(t)-2a_{-1}\tau(t)+a_1b^2{}_0\tau\ (t)-2\ a_0b_0\beta(t)+2\ a_1a_{-1}\beta(t)+k^2a_0-a_1b^2{}_0\beta(t)-4\ k^2a_{-1} +2a_1b_{-1}\tau(t)+4k^2a_1b_{-1}-2a_1b_{-1}\beta(t)+a_1^2b_{-1}\beta(t)+\ ka_0a_1b_0\alpha(t)$$

$$C_0 = 6k^2a_0b_{-1}+2a_0a_{-1}\beta(t)-2a_1b_0b_{-1}\beta(t)-2a_{-1}b_0\beta(t)-3a_{-1}b_0\tau(t)-3k^2a_{-1}b_0+ 3a_1b_0b_{-1}\tau(t)-3k^2a_1b_0b_{-1}+2a_1a_2b_0\beta(t)+\ 2a_1a_0b_{-1}\beta(t)-3ka_0a_{-1}\alpha(t)+\ 3ka_1a_0b_{-1}\alpha(t) -2a_0b_{-1}\beta(t)-a_0b^2{}_0\beta(t)\ +\ a^2{}_0b_0\beta(t)$$

$$C_{-3} = a^2{}_{-1}b_{-1}\beta(t)\ -\ a_{-1}b_{-1}^2\beta(t)$$

$$C_{-2} = -2a_{-1}b_0b_{-1}\beta(t) - k^2a_0b^2_{-1} - a_0b^2_{-1}\beta(t) - ka^2_{-1}b_0\alpha(t) + a^2_{-1}b_0\beta(t) + k^2a_{-1}b_0b_{-1}$$
$$+a_0b^2_{-1}\tau(t) + ka_2a_0b_{-1}\alpha(t) - a_{-1}b_0b_{-1}\tau(t) + 2a_{-1}a_0b_2\beta(t)$$

$$C_{-1} = -2ka^2_{-1}\alpha(t) - a_1b^2_{-1}\beta(t) + a_2{}^2\beta(t) + a^2{}_0b_{-1}\beta(t) + 2ka_{-1}a_1b_{-1}\alpha(t) + 2a_1b^2{}_{-1}\tau(t) - a_{-1}b_0^2\tau(t)$$
$$+ 2a_0a_{-1}b_0\beta(t) - a_{-1}b_0^2\beta(t) + k^2a_0b_{-1}b_0 + 2a_{-1}a_1b_{-1}\beta(t) - 2a_0b_{-1}b_0\beta(t) - 4k^2a_1b^2_{-1} + a_0b_{-1}b_0\tau(t)$$
$$-2a_{-1}b_{-1}\beta(t) - ka_0a_{-1}b_0\alpha(t) - 2a_{-1}b_{-1}\tau(t) - k^2a_{-1}b^2{}_0 + 4k^2a_{-1}b_{-1} + ka_0^2b_{-1}\alpha(t)$$

Equating to zero the coefficients of all powers of e^{ξ} yields a set of algebraic equations for $a_0, a_1, a_{-1}, b_{-1}, b_0, b_1, k\ \alpha(t), \beta(t), \tau(t)$. Solving the system of equations we obtain

Case-1 $a_0 = a_{0,}\ a_0 = 0,\ a_{-1} = a_0b_{0,}\ b_0 = b_0,\ b_{-1} = 0,\ \tau(t) = -k^2 - \beta(t),\ \alpha(t) = \dfrac{\beta(t)}{k}$ (15)

Case-2 $a_0 = 0\ \ a_1 = 1,\ a_{-1} = 0_{,} b_0 = 0,\ b_{-1} = b_1, \tau(t) = 2k^2 + \dfrac{\beta(t)}{2}, \alpha(t) = -4k$ (16)

Case-3 $a_0 = \dfrac{b_0 + \sqrt{b_0{}^2 - 4b_{-1}}}{2}, a_1 = 0_{,}\ a_{-1} = b_{-1,} b_0 = b_0,\ b_{-1} = b_{-1}, \tau(t) = -k^2 - \beta(t), \alpha(t) = 2k$ (17)

Case-4 $a_0 = \dfrac{b_0 - \sqrt{b_0{}^2 - 4b_{-1}}}{2},\ a_1 = 1_{,}\ a_{-1} = 0_{,} b_0 = b_0,\ b_{-1} = b_{-1}, \tau(t) = k^2 + \beta(t),\ \alpha(t) = -2k$ (18)

Case-5 $a_0 = a_{0,} a_1 = 1,\ a_{-1} = -b_0{}^2 + a_0b_{0,}\ b_0 = b_0,\ b_{-1} = 0, \tau(t) = -k^2, \alpha(t) = \dfrac{\beta(t)}{k}$ (19)

Case-6 $a_0 = a_{0,}\ a_1 = 1,\ a_{-1} = 0_{,}\ b_0 = b_0,,\ b_{-1} = b_{-1}$

$$\tau(t) = \frac{k^2\left(12\sqrt{2b_{-1}}a_0{}^4 + 7\sqrt{2}\,b_{-1}{}^{5/2} + 40\sqrt{2}\,b_{-1}{}^{5/2}a_0{}^2 + 45a_0{}^3b_{-1} + 2a_0{}^5 + 37a_0\right)}{10\sqrt{2}a_0{}^2b_{-1}^{3/2} + \sqrt{2}b_{-1}^{5/2} + 6\sqrt{2b_{-1}}a_0{}^4 + 2a_0{}^5 + 7a_0b_{-1}^2 + 15a_0{}^3b_{-1}}$$

$$\alpha(t) = -2k,\ \beta(t) = \frac{6k^2\left(2a_0b_{-1}{}^{3/2}\sqrt{2} + \sqrt{2b_{-1}}a_0^3 + 3a_0\right)}{4\sqrt{2b_{-1}}a_0^3 + 3\sqrt{2}a_0b_{-1}{}^{3/2} + 7a_0^2b_{-1} + 2a_0^4 + b_{-1}{}^2} \quad (20)$$

Substituting equation (15) to (20) into (13) yields

$$u_1(x,t) = \frac{a_0 + a_0b_0\exp\left[-kx + \int(k^2 + \beta(t))dt\right]}{\exp\left[kx - (k^2 + \beta(t))dt\right] + b_0} \quad (21)$$

$$u_2(x,t) = \frac{\exp\left[kx + \int(2k^2 + \frac{\beta(t)}{2})dt\right]}{\exp\left[kx + \int(2k^2 + \frac{\beta(t)}{2})dt\right] + b_{-1}\exp\left[-kx - \int(2k^2 + \frac{\beta(t)}{2})dt\right]} \quad (22)$$

$$u_3(x,t) = \frac{\frac{b_0 + \sqrt{b_0^2 - 4b_{-1}}}{2} + b_{-1}\exp\left[-kx + \int (k^2 + \beta(t))dt\right]}{\exp\left[kx - \int (k^2 + \beta(t))dt\right] + b_0 + b_{-1}\exp\left[-kx - \int (k^2 + \beta(t))dt\right]} \tag{23}$$

$$u_4(x,t) = \frac{\exp\left[kx + \int (k^2 + \beta(t))dt\right] + \frac{b_0 - \sqrt{b_0^2 - 4b_{-1}}}{2}}{\exp\left[kx + \int (k^2 + \beta(t))dt\right] + b_0 + b_{-1}\exp\left[-kx - \int (k^2 + \beta(t))dt\right]} \tag{24}$$

$$u_5(x,t) = \frac{\exp\left(kx - k^2 t\right) + a_0 + (b_0^2 + a_0 b_0)\exp\left(-kx + k^2 t\right)}{\exp\left(kx - k^2 t\right) + b_0} \tag{25}$$

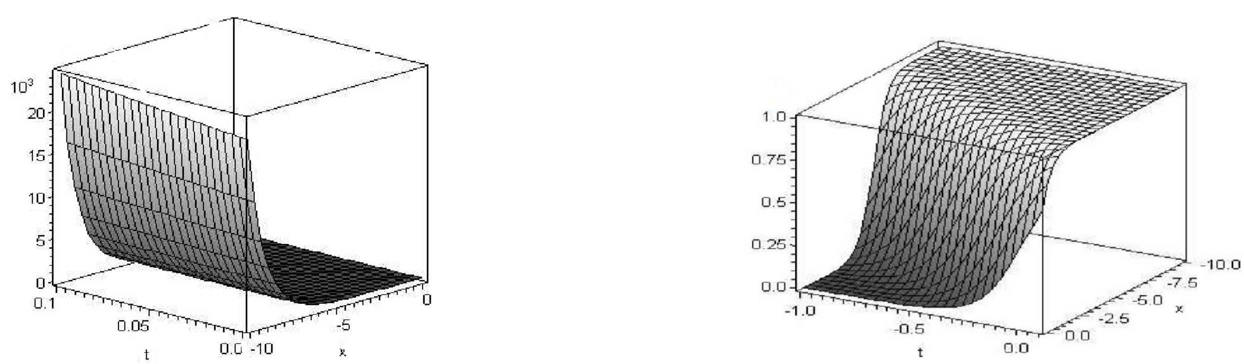
$$u_6(x,t) = \frac{\exp\left(kx + \int \tau\ (\mathrm{t})\mathrm{dt}\ \right) + a_0}{\exp\left(kx + \int \tau\ (\mathrm{t})\mathrm{dt}\ \right) - \sqrt{2b_{-1}} + b_{-1}\exp\left(-kx - \int \tau\ (\mathrm{t})\mathrm{dt}\ \right)} \tag{26}$$

Where $\tau(t) = \dfrac{k^2\left(12\sqrt{2b_{-1}}\mathrm{a}_0^{\,4} + 7\sqrt{2}\,b_{-1}^{\,5/2} + 40\sqrt{2}\,b_{-1}^{\,5/2}\mathrm{a}_0^{\,2} + 45\,\mathrm{a}_0^{\,3}\,b_{-1} + 2\mathrm{a}_0^{\,5} + 37\mathrm{a}_0\right)}{10\sqrt{2}\mathrm{a}_0^{\,2}b_{-1}^{3/2} + \sqrt{2}b_{-1}^{5/2} + 6\sqrt{2b_{-1}}\mathrm{a}_0^{\,4} + 2\mathrm{a}_0^{\,5} + 7\mathrm{a}_0 b_{-1}^2 + 15\mathrm{a}_0^{\,3}b_{-1}}$

3. NUMERICAL ILLUSTRATION:

(1) If we take $b_0 = 0$ we have $\mathrm{u}_{11}(\mathrm{x,t}) = a_0\exp\left(-kx + \int k^2 + \beta\ (\mathrm{t})\mathrm{dt}\ \right)$ (27)

(2) If we take $\alpha(t) = a$ is a constant $b_{-1} = 1$ and $\beta(t) = \dfrac{2ac - a^2}{4}$ in equation (22) where *c* is a constant then we have $\mathrm{u}_{21}(\mathrm{x,t}) = \dfrac{1}{2} - \dfrac{1}{2}\tanh\left[\dfrac{a}{4}(x - ct)\right]$ (28)



(a) (b)

Fig 1: (a) Solution of Eq. (27) with $a_0 = 1$, $k = 1$ and $\beta(t) = t$. (b) Solution of Eq. (28) with $a = 4$, $c = 5$.

(3) If we take $b_0 = 4$, $b_{-1} = 1$ and $k = 1$ in equation (23) we have

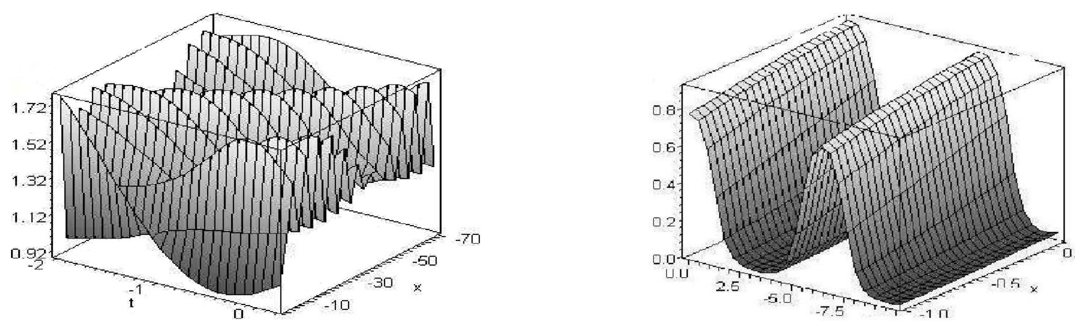
$$\mathrm{u}_{31}(\mathrm{x,t})=\frac{-1+\left(2+2\sqrt{3}\right)\cos ec\left[x-\int(1+\beta\ (\mathrm{t}))\mathrm{dt}\right]+\coth\left[x-\int(1+\beta\ (\mathrm{t}))\mathrm{dt}\right]}{4\cos ec\left[x-\int(1+\beta\ (\mathrm{t}))\mathrm{dt}\right]+2\coth\left[x-\int(1+\beta\ (\mathrm{t}))\mathrm{dt}\right]} \tag{29}$$

(4) If we take $b_0=0,\ b_{-1}=-5$ and $k=1$ in equation (24) we get

$$\mathrm{u}_{41}(\mathrm{x,t})=\frac{\cosh\left[x+\int(1+\beta\ (\mathrm{t}))\mathrm{dt}\right]+\sinh\left[x+\int(1+\beta\ (\mathrm{t}))\mathrm{dt}\right]-\sqrt{5}}{-4\cosh\left[x+\int(1+\beta\ (\mathrm{t}))\mathrm{dt}\right]+6\sinh\left[x+\int(1+\beta\ (\mathrm{t}))\mathrm{dt}\right]} \tag{30}$$

(5) If we take $b_0=2\ a_0=3/2$ in equation (25) we have

$$u_{51}(x,t)=\frac{2\tanh\left(kx-k^2t\right)+\frac{3}{2}\sec\left(kx-k^2t\right)}{1+\tanh\left(kx-k^2t\right)+2\sec\left(kx-k^2t\right)} \tag{31}$$

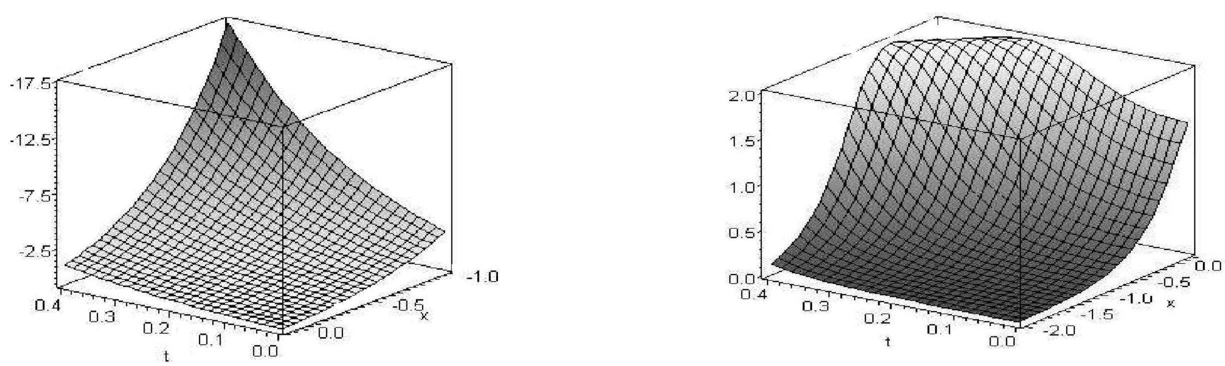


(a) (b)

Fig.2: (a) Solution of Eq. (29) with $\beta(\mathrm{t})=\cos t$ *(b) Solution of Eq. (30) with* $\beta(t)=-1+3\sin t$.

(6) If $a_0=0$ in equation (26) we have

$$u_{61}(x,t)=\frac{\exp\left(kx+7k^2t\right)}{\exp\left(kx+7k^2t\right)-\sqrt{2b_{-1}}+b_{-1}\exp\left(-kx-7k^2t\right)} \tag{32}$$



(a) (b)

Fig. 3: (a) Solution of Eq. (31) with $k = 2$. (b) Solution of Eq. (32) with $k = 1$ and $b_{-1} = 2$.

4. CONCLUSION

The Nonlinear Evolution equation with variable coefficients is investigated by Exp-function method. The generalized travelling wave solutions of this equation are obtained with the help of symbolic computation. From these results, we can see that the Exp-function method is one of the most effective methods to obtain exact solutions.

Finally, it is worthwhile to mention that the Exp-function method can also be extended to other nonlinear evolution equations with variable coefficients, such as the mKdV equation, the (3 +1)-dimensional Burgers equation, the generalized Zakharov-Kuznetsov equation and so on. The Exp-function method is a promising and powerful new method for nonlinear evolution equations.

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